FUZZY MEAN RESIDUAL LIFE ORDERING OF FUZZY RANDOM VARIABLES

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ABSTRACT:

It is evident that the shape of the failure rate function plays an inevitable role in repair and replacement strategies, the mean residual life function is more relevant as the same summarizes the entire residual life functions.

In this paper we formulate different fuzzy partial ordering results related to mean residual life order and proportional mean residual life model with some characterization results. Some properties of the up mean residual life model have been obtained along with the closure of the up mean residual life order under mixture type operation.

We consider fuzzy random variables to capture the mean residual function and s how that redundancy at the component level is not superior to that at the system level. Even when the lifetimes are original and the spare components are i.i.d., though the result holds for usual stochastic order. The up mean residual life order is characterized in terms of the DMRL class. We capture some of the characterizations of the DMRL class.

Keywords: Fuzzy random variables, Hazard rate order, Stochastic order, Mean residual life order, DMRL and fuzzy up mean residual order.

INTRODUCTION

In life testing situations, the mean additional life time given that a component has survived until time t is a function of t, called the mean residual life. We consider the special case of mean residual life interms of fuzzy random variables. More specifically, if the fuzzy random variables X represents the life of a component, then the mean residual life is given by $m_\alpha(t) = E[X_\alpha - t / X_\alpha > t]$.

The mean residual life has been employed in life length by various authors, e.g. Bryson and siddigui (1969), Hollander and Proschan (1975) and Muth (1977). Limiting properties of the mean residual life have been studied by Meillijson (1972), Balkema and de Hann (1974), and more recently by Bradley and Gupta [2], (2002). A smooth estimator of the mean residual life is given by chaubey and Sen (1999).

It is well known that the failure rate function can be expressed quite well in terms of mean residual life and its derivative. However, the inverse problem – namely that of expressing the mean residual life in terms of the failure rate typically involves an integral of the complicated expressions.

1. PRELIMINARIES

Let $\bar{F}\ (t) = P\ (X > t)$ be the survival function of a random variable X having finite mean μ. Then t mean residual life (mrl) m(t) of X is defined as

m(t) = E[X – t / X > t] for t < t* and t = 0 otherwise, where t* = sup{t: $\bar{F}\ (t)$ > 0}.

If P(X ≥ 0) = 1 then μ is finite and m(0) = μ. Also m(t) < ∞ for 0 ≤ t < ∞.

If t* = ∞ then we have m(t) = $\frac{\int_t^\infty \bar{F}(x)dx}{\bar{F}(t)}$

In this paper, we consider only non-negative fuzzy random variables, although most of the results can be proved for fuzzy random variables with support in (0, 1].

Note that, although m(t) ≥ 0 for all t, every non-negative function does not respect the mean residual life function corresponding to some random variable. In fact, a function m(.) represents a mean residual life function of some non-negative random variable with an absolutely continuous distribution function iff it possesses the following properties by Bhattacharjee [1], Shaked and Shanthikumar[8] and JEL Piriyakumar and N. renganathan[3]

i) 0 ≤ m(t) < ∞ for all t ≥ 0.

ii) m(0) ≥ 0

iii) m(t) is continuous

iv) m(t) + t is increasing on [0, ∞]

v) when these exists a t_0 such that m(t_0) = 0 then m(t) = 0 for all t ≥ t_0. Otherwise, when there does not exist such a t_0 with m(t_0) = 0 then $\int_0^\infty \frac{1}{m(t)} dt = \infty$

The smaller the mean residual life function, the smaller the variable X should be in some stochastic sense. This statement gives the motivation behind the mean residual life order defined as follows:

Definition 1.1:

Let X and Y be two random variables with mean residual life functions m_X and m_Y respectively. Then X is said to be smaller than Y in the mean residual life order, denoted by X$\leq_{mrl}$Y if m_X(t) ≤ m_Y(t) for all t.

X$\leq_{mrl}$Y iff $\frac{\int_x^\infty \overline{F_X}(u)du}{\int_x^\infty \overline{F_Y}(u)du}$ decreasing in x

over $\{x\colon \int_x^\infty \overline{F_Y}(u)du > 0\}$ where $\overline{F_X}(.)$ denotes the survival function of the random variable Z.

Definition 1.2:

X is said to be mean residual life aging faster than Y if $\frac{m_X(t)}{m_Y(t)}$ is increasing in t ≥ 0 (or) ultimately mean residual life aging faster than Y if above holds for sufficiently large t.

2. FUZZY RANDOM VARIABLES

The concept of fuzzy random variable was introduced by Kwakernaak [5,6] and Puri and Ralescu[8]. A fuzzy random variable is just a random variable that takes on values in a space of fuzzy sets. The outcomes of Kwakernaak's fuzzy random variables are fuzzy real subsets and the extreme points of their α-cuts are classical random variables. Fuzzy random variables are mathematical descriptions for fuzzy stochastic phenomena, but only one time descriptions.

Definition 2.1:

Let (Ω , 𝔸, P) be probability space. A fuzzy set valued mapping X: Ω →F(R) is called a fuzzy random variable if for each B∈$\mathcal{B}$ and for each α∈(0, 1]. $X_\alpha^{-1}(B) = \{\omega\epsilon\Omega;\ X_\alpha(\omega) \cap B \neq \varphi\} \in$ 𝔸.

3. FUZZY MEAN RESIDUAL LIFE ORDERING

In this section the concept of fuzzy mean residual life functions are introduced. The conceptualization is accomplished using resolution identity. It is possible to construct a closed fuzzy number from a family of closed intervals. Using this technique the formulation of fuzzy probability distribution functions, fuzzy mean residual life functions are introduced. It is established that under certain assumptions the relative fuzzy hazard rate ordering leads to the corresponding ultimate fuzzy mean residual life ordering.

Let X be a non-negative random variable with distribution function (fuzzy) F(x) and density function (fuzzy) f(x). Let F(x) denote the failure distribution of X,

$r_F(x) = \frac{f(x)}{\bar{F}(x)}$ denote the hazard rates and $\bar{F}(x) = 1 - F(x)$ denote the survival function X.

Definition 3.1:

Let X be a non-negative fuzzy random variable with survival function $\bar{F}_\alpha$ and a finite mean μ. The α-level mean residual life of X_α at t, for α∈(0, 1] is defined as $m_\alpha(t) = \begin{cases} E[X_\alpha - t/X_\alpha > t; for\ t < t^* \\ 0\ , otherwise \end{cases}$ where t^* = sup{t: $\bar{F}\ (t) > 0$}.

Definition 3.2:

Let X be anon-negative fuzzy random variable with an absolutely continuous distribution function F. The α-level hazard rate of X is defined as

$r_\alpha(t) = \frac{f_\alpha(t)}{\overline{F_\alpha}(t)}$; t∈R, α∈(0, 1].

Definition 3.3:

Let X and Y be two non-negative fuzzy random variables with α-level mean residual functions m_α and l_α respectively such that m_α(t) ≤ $l_\alpha(t)$ for all t and for each α∈(0, 1]. Then X is said to be smaller than Y in fuzzy mean residual life order. Symbolically, it is denoted as X$\leq_{fmrl}$Y.

Definition 3.4:

Let X and Y be two non-negative fuzzy random variables with survival functions $\overline{F_\alpha}(x)$ and $\overline{G_\alpha}(x)$ respectively. Then X$\leq_{fmrl}$Y iff $\frac{\int_t^\infty \overline{G_\alpha}(u)du}{\int_t^\infty \overline{F_\alpha}(u)du}$ increases in t over $\{t\colon \int_t^\infty \overline{F}_\alpha(u)du > 0\}$ for each α∈(0, 1].

The following result, due to JEL Piriyakumar and A. Yamuna[4], connects the fuzzy mean residual life order with hazard rate order.

Theorem 3.5:

Let X and Y be two non- negative fuzzy random variables with α-level mean residual life functions m_α and l_α respectively. Suppose that $\frac{m_\propto(t)}{l_\propto(t)}$ increases in t and for each α ∈ (0, 1]. Then if X $\leq_{fmrl}$ Y then it follows that X $\leq_{fhr}$ Y.

4. SOME NEW RESULTS

This section presents some new results concerning the fuzzy mean residual life order and some characterization results.

Definition 4.1:

A non-negative fuzzy random variable X is said to be smaller than another fuzzy random variable Y in the fuzzy up mean residual order, denoted by X$\leq_{fmrl\uparrow}$Y if $X_\alpha - x \leq_{fmrl\uparrow} Y_\alpha$ for all x ≥ 0 and for each α∈(0, 1].

It is to be noted that $m_{X_{\alpha-x}}(t) = m_{X_\alpha}(x+t)$ for each α∈(0, 1]. Symbolically,

$X_{\alpha x}$ = $[X_\alpha - x/X_\alpha > x]$.

Definition 4.2:

Let X be a non-negative fuzzy random variable with survival function $\bar{F}_\alpha$ and a finite mean μ. The α-level up mean residual life of $X_{\alpha x}$ at t, for each α∈(0, 1] is defined as

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$m_{X_{\alpha x}}(t) = \begin{cases} E[(X_\alpha - x) - t / X_\alpha - x > t; for\ t < t^* \\ 0\ , otherwise \end{cases}$ where $t^* = \sup\{t: \bar{F}(t) > 0\}$.

Clearly $m_{X_{\alpha x}}(t)$ is differentiable over {t : P($X_{\alpha x}$ > t) > 0}

And we get $\int_t^\infty \overline{F_{X_{\alpha x}}}(u)du = \frac{1}{\overline{F_{X_\alpha}}(x)} \int_{t+x}^\infty \bar{F}_\alpha(u)du$ and $m_{X_{\alpha x}}(t) = m_{X_\alpha}(t+x)$ for each α∈(0, 1].

Also, X $\leq_{fmrl\uparrow}$ Y iff α

Proposition 4.3:

X$\leq_{fmrl\uparrow}$ X iff X is DMRL.

Proof:

By definition, X$\leq_{fmrl\uparrow}$ X

⇔ $X_\alpha - x \leq_{fmrl} X_\alpha$ for x ≥ 0 and for each α∈(0, 1].

⇔ $[X_\alpha - x / X_\alpha > x]$ ≥ hmrl $[X_\alpha - x' / X_\alpha > x']$

whenever x' ≥ x ≥ 0

⇔ $\cup_{\alpha\epsilon(0,1]}[X_\alpha - x / X_\alpha > x] \geq hmrl \cup_{\alpha\epsilon(0,1]}[X_\alpha - x' / X_\alpha > x']$ $whenever\ x' \geq x \geq 0$

⇔ X is DMRL (by definition of DMRL)

Proposition 4.4:

If X $\leq_{fmrl\uparrow}$ Y and Y $\leq_{fmrl\uparrow}$ Z, then X $\leq_{fmrl\uparrow}$ Z.

Proof:

X $\leq_{fmrl\uparrow}$ Y ⇒ $X_\alpha - x \leq_{fmrl} Y_\alpha$ for all x ≥ 0, for each α∈(0, 1]. (1)

Similarly Y $\leq_{fmrl\uparrow}$ Z ⇒ $Y_\alpha - y \leq_{fmrl} Z_\alpha$ (2)

To prove X $\leq_{fmrl\uparrow}$ Z

By (1),

$X_\alpha - x - y \leq_{fmrl} Y_\alpha - y$ for x, y ≥ 0

$\leq_{fmrl} Z_\alpha$

by (2)

⇒ $X_\alpha - x' \leq_{fmrl} Z_\alpha$

where x' = x – y

⇒ $\cup_{\alpha\epsilon(0,1]}[X_\alpha - x']$ ≤ fmrl $\cup_{\alpha\epsilon(0,1]} Z_\alpha$

⇒ X $\leq_{fmrl\uparrow}$ Z.

Proposition 4.5:

If X $\leq_{fmrl\uparrow}$ Y and Y $\leq_{fmrl\uparrow}$ X then X $=_d$ Y.

Proof:

It is obvious from the definition that up fuzzy mean residual life order implies fuzzy mean residual life order.

Now, we give some condition under which FMRL order implies FMRL↑ order.

Theorem 4.6:

If X $\leq_{fmrl}$ Y and either x or Y is DMRL, then X $\leq_{fmrl\uparrow}$ Y.

Proof:

We know that X $\leq_{fmrl\uparrow}$ Y iff , X $\leq_{fmrl\uparrow}$ Y iff $\frac{\int_t^\infty \overline{F_\alpha}(x+u)du}{\int_t^\infty \overline{G_\alpha}(u)du}$ is decreasing in t over $\{t: \int_t^\infty \bar{F}_\alpha(u)du > 0\}$, for all x ≥ 0 and for each α∈(0, 1].

Now, we can write

$$\frac{\int_t^\infty \overline{F_\alpha}(x+u)du}{\int_t^\infty \overline{G_\alpha}(u)du} = \left(\frac{\int_t^\infty \overline{F_\alpha}(x+u)du}{\int_t^\infty \overline{F_\alpha}(u)du}\right) \left(\frac{\int_t^\infty \overline{F_\alpha}(u)du}{\int_t^\infty \overline{G_\alpha}(u)du}\right)$$

(3)

We see that the first factor on the RHS of (3) is decreasing in t if X is DMRL and the second factor is decreasing in t, as X $\leq_{fmrl}$ Y.

Hence, $\frac{\int_t^{\infty} \overline{F_\alpha}(x+u)du}{\int_t^{\infty} \overline{G_\alpha}(u)du}$ is decreasing in t for x ≥ 0 and for each α∈(0, 1].

If Y is DMRL, then proof is similar.

∴ by proposition 4.3 and by 4.4 we see that X $\leq_{fmrl\uparrow}$ Y.

Remark:

The non-negative random variable X is DMRL if and only if [X – t / X > t] $\geq_{hmrl}$ [X – t' / X > t'].

Theorem 4.7:

Let X and Y be two non-negative fuzzy random variables with α-level mean residual life functions m_α and l_α respectively. Suppose that $\frac{m_\alpha(t)}{l_\alpha(t)}$ is increasing in t ≥ 0 and for each α∈(0, 1]. Then X $\leq_{hr}$ Y if X $\leq_{fmrl\uparrow}$ Y.

Proof:

$m_{X_{\alpha x}}(t) = \begin{cases} E[(X_\alpha - x) - t/X_\alpha - x > \; t; for\; t < \; t^* \\ 0\, , otherwise \end{cases}$ where t^* = sup{t: $\bar{F}\,(t) > 0$} and α∈(0, 1].

Clearly $m_{X_{\alpha x}}(t)$ is differentiable over {t : P($X_{\alpha x}$ > t) > 0} and that if X has the α-level fuzzy hazard rate function r_α then

$r_\alpha(t) = \frac{m_{\alpha'}(t)+1}{m_\alpha(t)}$ where $m_{\alpha'}$ (t) denote the derivative of m_α(t).

Similarly, if Y has the α-level fuzzy hazard rate function q_α, then $q_\alpha(t) = \frac{l_{\alpha'}(t)+1}{l_\alpha(t)}$.

By stipulation $\frac{m_\alpha(t)}{l_\alpha(t)}$ increases in t for each α∈(0, 1] and X $\leq_{fmrl\uparrow}$ Y.

This shows that $m_\alpha(t) \leq l_\alpha(t)$ for all t and for each α∈(0, 1].

Now, $r_\alpha(t) = \frac{m_{\alpha'}(t)}{m_\alpha(t)} + \frac{1}{m_\alpha(t)}$

$\geq \frac{l_{\alpha'}(t)}{l_\alpha(t)} + \frac{1}{l_\alpha(t)}$

$= q_\alpha(t)$

i.e) $r_\alpha(t) \geq q_\alpha(t)$ for all t and for each α∈(0, 1].

Hence, $\bigcup_{\alpha\epsilon(0,1]} \alpha\, r_\alpha(t) \geq \bigcup_{\alpha\epsilon(0,1]} \alpha\, q_\alpha(t)$

i.e) r(t) ≥ q(t) ⇒ X $\leq_{hr}$ Y

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